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EFFECT OF RESOURCE BIOMASS ON STAGE STRUCTURED PREDATOR PREY SYSTEM HAVING HOLLING TYPE III FUNCTIONAL RESPONSE

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ABSTRACT

This paper deals with predator-prey model having Holling type III functional response. The prey population is stage structured consisting of immature and mature stages and the predator population is influenced by the resource biomass. Dynamical behaviors such as positivity, boundedness, stability, bifurcation of the model are studied analytically using theory of differential equations. Computer simulations are carried out to prove the analytical result. It is noted that influence of resource biomass on the predator population may lead to the extinction of predator at a lesser value of maturity time in comparison to the absence of resource biomass.

KEYWORD: Prey-Predator Model, Stability, Stage-Structure, Functional Response, Resource Biomass

1. INTRODUCTION

One of the most powerful tools for understanding the predator prey relationship in ecology is to understand the dynamic relationship between predator and their prey. Many researchers have studied the system in depth [1, 7, 14, 17, 18, 19, and 22]. The Stage structure population where the individual member have a life history that takes them through two stages, immature and mature have received much attention in mathematical modeling of ecological system. In all these studies, the maturity age is represented by time delay thus the resulting in the system of retarded functional differential equations. Zang ET. al., [26] in their paper investigated the behavior of predator -prey population with prey as structured population. Song and Chen [20] studied a two species competitive stage structured population model with harvesting for prey. They obtained stability conditions and threshold of harvest effort for population survival.

It may be pointed out here, that most of the above studies are based on the traditional predator-prey models with either prey or predator stage-structured or both. In nature, there are many cases where predator population dynamics is influenced by the presence of an additional resource (which may or may not be a secondary prey). This can be illustrated with the example: stone martins, relative of the weasel, are extremely fierce and dangerous predators, and often take prey like squirrels, pike, voles, hares, etc. [2]. They live in forest, especially evergreen ones and spend much of their time up on trees, jumping from one place to another, climbing up and down and rarely reaching the ground. They build a den in an abandoned hole in a tree (resource biomass) and hence produce an adverse effect on the growth of the trees. This example shows a relationship between resource biomass and predator population, although the predator feeds on prey only. Consequences of moose acting as additional prey in a Wolf–Caribou system have been described by Bergerud [4] and Bergerud *et al*. [5]. Freedman and Shukla [9] analyzed a predator-prey system where the resource dynamics affects the predator-prey system. Freedman *et al.* [8] analyzed a ratio-dependent predator-prey model where the predator population is influenced by the presence of a resource.

Waryano Sunaryo, et. al., [21] studied an ecological model with a tri-trophic food chain composed of a classical Lotka-Volterra functional response for prey and predator, and a Holling type-III functional response for predator and super-predator. Agarwal and Pathak [3] studied the effect of harvesting on dynamics of prey predator model with holling type III functional response.

It may be noted from the above investigations that behavior of resource biomass on predator- prey stage structure model with Holling type III functional response is not studied yet. Therefore in this paper our focus is to model the effect of resource biomass on predator prey population with Holling type III functional response where prey population is stage structured.

2. MATHEMATICAL MODEL

In this paper Holling type III predator functional response on predator – prey – resource model with stage – structure for prey as the system of following differential equations:

$$\dot{x}_i(t) = \alpha x_m(t) - \gamma x_i(t) - \alpha e^{-\gamma\tau} x_m(t-\tau),$$

$$\dot{x}_m(t) = \alpha e^{-\gamma\tau} x_m(t-\tau) - \beta x_m^2(t) - \frac{c x_m^2(t) y(t)}{x_m^2(t) e_1 + 1}, \tag{2.1}$$

$$\dot{y}(t) = y(t)\left(-d(r(t)) + \frac{f x_m^2(t)}{1 + e_1 x_m^2(t)} \right),$$

$$\dot{r}(t) = r(t)\left(1 - \frac{r(t)}{L(y(t))} \right),$$

$$x_m(t) = \phi_m(t) \ge 0, \ -\tau \le t < 0 \ \text{and} \ x_i(0) > 0, \ y(0) > 0, \ r(0) > 0,$$

Where $x_i(t)$ and $x_m(t)$ are the densities of immature and mature prey populations respectively, $y(t)$ is the density of the predators and $r(t)$ is the density of the resource biomass.

In mathematical model (2.1) all the parameters are positive under the following assumptions.

H1. α is the proportionality constant for immature prey population to mature prey population, τ is delay period at which immature population transfer to mature prey population, γ is death rate of immature prey population and β be the intra specific interaction.

The term $\alpha e^{-\gamma\tau} x_m(t-\tau)$ describes that immature prey population born at time $(t-\tau)$ and surviving at the time t and further covert immature to mature prey population. c, e_1 and f are constants and are defined as the capturing rate of predator, efficiency rate of predator on prey and conversion rate of predator, respectively.

H2. $d(r) = d_0 - d_1 + d_1 e^{-r}$ Is the death rate of predator, which decreases as the density of resource biomass increases,

$$d(0) = d_0 > 0, \ d'(r) < 0, \ r > 0.$$

$L(y) = L_0 - L_1 + L_1 e^{-y}$, Is the carrying capacity of the resource and it decreases as y increases. $L(0)$ is the carrying capacity of the resource in the absence of predator y.

Where,

$L(0) = L_0 > 0,\ L_1 \in (0, L_0)$, And

$\lim_{y\to\infty} L(y) = (L_0 - L_1) = L_\infty \text{ (say) } > 0.$

Now for continuity of initial conditions, we require

$$x_i(0) = \int_{-\tau}^{0} \alpha e^{\gamma s} \phi_m(s) ds, \tag{2.2}$$

The total surviving immature population from the observed births on $-\tau \le t < 0$.

With the help of equation (2.2), the solution of the first equation of system (2.1) can be written in terms of solution for $x_m(t)$ as:

$$x_i(t) = \int_{t-\tau}^{t} \alpha e^{-\gamma(t-s)} x_m(s) ds. \tag{2.3}$$

Equations (2.2) and (2.3) suggest that, mathematically no information on the past history of $x_i(t)$ is needed for the system (2.1), because the properties of $x_i(t)$ can be obtained from (2.2) and (2.3) if we know the properties of $x_m(t)$.

Therefore we need only to consider the following system of equations,

$$\dot{x}_m(t) = \alpha e^{-\gamma\tau} x_m(t-\tau) - \beta x_m^2(t) - \frac{c x_m^2(t) y(t)}{x_m^2(t) e_1 + 1},$$

$$\dot{y}(t) = y(t)\left(-d(r(t)) + \frac{f x_m^2(t)}{1 + e_1 x_m^2(t)}\right), \tag{2.4}$$

$$\dot{r}(t) = r(t)\left(1 - \frac{r(t)}{L(y)}\right),$$

$x_m(t) = \phi_m(t) \ge 0,\ -\tau \le t < 0$, And $y(0) > 0,\ r(0) > 0$.

3. BOUNDEDNESS OF SOLUTIONS

Lemma 3.1

The set $R = \left\{(x_m, y, r) : 0 \le x_m \le \frac{\alpha e^{-\gamma\tau}}{\beta}, 0 \le \frac{x_m}{c} + \frac{y}{f} \le \frac{\alpha^2 e^{-2\gamma\tau}}{c\beta\delta}, 0 \le r \le L_0\right\}$ attract all solutions initiating in the interior of the positive region, Where $\delta = \min\left\{\alpha e^{-\gamma\tau}, d_0 - d_1\right\}$.

Proof:

Consider the following equation,

$$\dot{x}_m(t) = \alpha e^{-\gamma\tau} x_m(t-\tau) - \beta x_m^2(t) - \frac{c x_m^2(t) y(t)}{x_m^2(t) e_1 + 1}$$

We obtain that,

$$x_m(t) \le \frac{\alpha e^{-\gamma\tau}}{\beta}.$$

Now form fist and second equations of system (2.4), we obtain the following

$$\frac{d}{dt}\left(\frac{x_m}{c} + \frac{y}{f}\right) \le \frac{\alpha e^{-\gamma\tau} x_m(\mathrm{t}-\tau)}{c} - \frac{\beta x_m{}^2}{c} - \frac{y(d_0 - d_1)}{f},$$

$$\le \frac{\alpha^2 e^{-2\gamma\tau}}{\beta c} - \frac{\beta x_m{}^2}{c} - \frac{y(d_0 - d_1)}{f},$$

$$\le \frac{\alpha^2 e^{-2\gamma\tau}}{\beta c} - \alpha e^{-\gamma\tau} \frac{x_m}{c} - \frac{y(d_0 - d_1)}{f},$$

$$\le \frac{\alpha^2 e^{-2\gamma\tau}}{\beta c} - \min\left\{\alpha e^{-\gamma\tau}, (d_0 - d_1)\right\}\left(\frac{x_m}{c} + \frac{y}{f}\right),$$

This implies that $0 \le \frac{x_m}{c} + \frac{y}{f} \le \frac{\alpha^2 e^{-2\gamma\tau}}{c\beta\delta}$ where $\delta = \min\left\{\alpha e^{-\gamma\tau}, (d_0 - d_1)\right\}$.

Now from the third equation of system (2.4), we obtain

$$\dot{r}(t) \le r(t)\left(1 - \frac{r(t)}{L_0}\right).$$

This implies that, $\lim_{t\to\infty} \sup r(t) \le L_0$.

This completes the proof of the lemma.

4. EQUILIBRIUM POINTS AND STABILITY ANALYSIS:

For this system there exist only five positive equilibrium points which are given as:

$E_0(0,0,0)$, $E_1(x_{m1}, 0, 0)$, $E_2(x_{m2}, \mathrm{y}_2, 0)$, $E_3(x_{m3}, 0, r_3)$ and $E^*(x_m^*, y^*, r^*)$.

Where,

$$x_{m1} = \frac{\alpha e^{-\gamma\tau}}{\beta}, \quad x_{m3} = \frac{\alpha e^{-\gamma\tau}}{\beta} \text{ and } r_3 = L_0.$$

The existence of equilibrium points $E_0(0,0,0)$, $E_1(x_{m1}, 0, 0)$ and $E_3(x_{m3}, 0, r_3)$ are obvious.

The existence of point $E_2(\mathrm{x}_{m2}, \mathrm{y}_2, 0)$ is given by the equations,

$$\alpha e^{-\gamma\tau} - \beta x_{m2}(t) - \frac{c x_{m2}(t) y_2(t)}{x_{m2}^2(t) e_1 + 1} = 0, \quad -d_0 + \frac{f x_{m2}^2(t)}{1 + e_1 x_{m2}^2(t)} = 0.$$

From these equations we obtain that the point $E_2(x_{m2}, y_2, 0)$ exist if following conditions hold, $f > d_0 e_1$ and $\frac{\alpha e^{-\gamma\tau}}{\beta} > c(\eta^2 + 1)\eta$, where $\eta^2 = \frac{d_0}{f - d_0 e_1}$.

Now equilibrium point $E^*(x_m^*, y^*, r^*)$ exist if the system of equations,

$$\alpha e^{-\gamma\tau} - \beta x_m(t) - \frac{c x_m(t) y(t)}{x_m^2(t) e_1 + 1} = 0,$$

$$-d(r(t)) + \frac{f x_m^2(t)}{1 + e_1 x_m^2(t)} = 0,$$

$$1 - \frac{r(t)}{L(y)} = 0,$$

Has a positive solution. From second and third equations of above equations we get,

$$x_m^2 = \frac{d(L(y))}{\{f - e_1 d(L(y))\}} = \pi(y), \qquad (say).$$

Using this function in first equation, now we take,

$$F(y) = \left(\beta + \frac{c\pi(y)}{\pi(y) e_1 + 1}\right)^2 \pi(y) - \alpha^2 e^{-2\gamma\tau},$$

then,

(i). $F(0) = \beta^2 \pi(0) - \alpha^2 e^{-2\gamma\tau} < 0$, as $\pi(0) < \frac{\alpha^2 e^{-2\gamma\tau}}{\beta^2}$ and

(ii). $F(K) = \left(\beta + \frac{c\pi(K)}{\pi(K) e_1 + 1}\right)^2 \pi(K) - \alpha^2 e^{-2\gamma\tau} > 0$, where K is the maximum value of y.

Thus there will exist y^* such that $F(y^*) = 0$ for $y^* \in (0, K)$.

Now the sufficient condition for uniqueness of $E^*(x_m^*, y^*, r^*)$ is given as,

$$\frac{dF(y)}{d(y)} > 0.$$

The Dynamic behavior of the equilibrium points can be checked by the Jacobian matrix,

$$J(\mathrm{E}) = \begin{pmatrix} \alpha e^{-(\gamma+\lambda)\tau} - 2\beta x_m - \dfrac{2cx_m y}{(1+\mathrm{e}_1\, x_m^2)^2} - \lambda & -\dfrac{cx_m^2}{(1+\mathrm{e}_1\, x_m^2)} & 0 \\ \dfrac{2\, fx_m y}{(1+\mathrm{e}_1\, x_m^2)^2} & -d(r) + \dfrac{fx_m^2}{(1+\mathrm{e}_1\, x_m^2)} - \lambda & -yd'(r) \\ 0 & \dfrac{r^2 L'(y)}{[L(\mathrm{y})]^2} & 1 - \dfrac{2r}{L(\mathrm{y})} - \lambda \end{pmatrix}$$

Equilibrium point $E_0(0,0,0)$ is trivial and its characteristic roots are given by equation $\lambda_1 = \alpha e^{-(\gamma+\lambda)\tau}$, $\lambda_2 = -d_0$ and $\lambda_3 = 1$, showing equilibrium point $E_0(0,0,0)$ is unstable in direction of $x_m - r$ and stable in the direction of y , and hence saddle point.

The characteristic equation for $E_1\left(x_{m1}, 0, 0\right)$ is given as,

$$\left(\alpha e^{-(\gamma+\lambda)\tau} - 2\beta x_1 - \lambda\right)\left(-d_0 + \frac{fx_1^{\,2}}{(1+\mathrm{e}_1\mathrm{x}_1^{\,2})} - \lambda\right)(1-\lambda) = 0$$

Then the characteristic roots of this equation is given by equation $\lambda = \alpha e^{-(\gamma+\lambda)\tau} - 2\beta x_1$, $-d_0 + \dfrac{fx_1^{\,2}}{(1+\mathrm{e}_1\mathrm{x}_1^{\,2})}$ and 1, as $-d_0 + \dfrac{fx_1^{\,2}}{(1+\mathrm{e}_1\mathrm{x}_1^{\,2})} \geq 0$ it is unstable in direction $y - r$ and as $\mathrm{Re}\lambda < 0$ the eigenvalue $\alpha e^{-(\gamma+\lambda)\tau} - 2\beta x_1 < 0$ so it is stable in direction x_m , so $E_1\left(x_{m1}, 0, 0\right)$ is saddle point.

The characteristic equation for the point $E_2(\mathrm{x}_{m2}, \mathrm{y}_2, 0)$ is given as,

$$\left(\lambda^2 + X_1\lambda + X_2 + e^{-\lambda\tau}\left\{X_3\lambda + X_4\right\}\right)(\lambda - 1) = 0,$$

Then the eigenvalue in the direction r is 1. So it is unstable in this direction and others eigenvalues are given by the following equation:

$$\left(\lambda^2 + X_1\lambda + X_2 + e^{-\lambda\tau}\left\{X_3\lambda + X_4\right\}\right) = 0,$$

Where,

$$X_1 = 2\beta x_{m2} + \frac{2cx_{m2}y_2}{(1+\mathrm{e}_1\mathrm{x}_{m2}^{\;2})^2} + d_0 - \frac{fx_{m2}^{\;2}}{(1+\mathrm{e}_1\mathrm{x}_{m2}^{\;2})},$$

$$X_2 = \frac{2\, fcx_{m2}^{\;3}y_2}{(1+\mathrm{e}_1\mathrm{x}_{m2}^{\;2})^3},$$

$X_3 = -\alpha e^{-\gamma\tau}$, And

$$X_4 = -\alpha e^{-\gamma\tau}\left(d_0 - \frac{fx_{m2}^{\;2}}{(1+\mathrm{e}_1\mathrm{x}_{m2}^{\;2})}\right),$$

The stability analysis of this point seen by the theorem (4.1), so this point is saddle point which is stable only in the direction $x_{m2}-y_2$.

Now for the point $E_3\left(x_{m3},0,r_3\right)$ characteristic equation is given as,

$$\left(\alpha e^{-(\gamma+\lambda)\tau}-2\beta x_1-\lambda\right)\left(-d_0+\frac{fx_1^{\,2}}{(1+e_1x_1^{\,2})}-\lambda\right)(1-\lambda)=0$$

Then the eigenvalue in the direction of y is given as $-d_0+\frac{fx_{m3}^{\,2}}{(1+e_1x_{m3}^{\,2})}>0$ is positive and in the direction of $x_{m3}-r_3$ the eigenvalues are negative, the eigenvalue corresponding to x_{m3} direction is given by equation $\lambda=\alpha e^{-(\gamma+\lambda)\tau}-2\beta x_{m3}$, putting the value of $x_{m3}=\frac{\alpha e^{-\gamma\tau}}{\beta}$ it become $\lambda=\alpha e^{-\gamma\tau}(e^{-\lambda\tau}-2)$.

Suppose that $\operatorname{Re}\lambda\geq 0$ then we calculate the real part of this, we get $\operatorname{Re}\lambda=\alpha e^{-\gamma\tau}(e^{-\tau\operatorname{Re}\lambda}\cos(\tau\operatorname{Im}\lambda)-2)<0$ a contradiction. Hence $\operatorname{Re}\lambda<0$ and in the direction of r_3 eigenvalue is 1, so this point is saddle point.

Characteristics equation for the point $E^*(x_m^*,y^*,r^*)$ is given as:

$$\phi(\lambda,\tau)=\lambda^3+B_1\lambda^2+B_2\lambda+B_3+e^{-\lambda\tau}\left\{B_4\lambda^2+B_5\lambda+B_6\right\}=0, \tag{4.1}$$

Where B_1, B_2, B_3, B_4, B_5 and B_6 given as,

$$B_1=\left\{d(r^*)-1+\frac{2r^*}{L(y^*)}+2\beta x_m^*+\frac{2cx_m^*y^*}{(1+e_1x_m^{*2})^2}-\frac{fx_m^{*2}}{1+e_1x_m^{*2}}\right\},$$

$$B_2=\left(2\beta x_m^*+\frac{2cx_m^*y^*}{(1+e_1x_m^{*2})^2}\right)\left(-d\left(r^*\right)+\frac{fx_m^{*2}}{1+e_1x_m^{*2}}\right)+\left(1-\frac{2r^*}{L\left(y^*\right)}\right)\left(2\beta x_m^*+\frac{2cx_m^*y^*}{(1+e_1x_m^{*2})^2}-d\left(r^*\right)+\frac{fx_m^{*2}}{(1+e_1x_m^{*2})}\right)$$
$$+\frac{2fcx_m^{*2}y^*}{(1+e_1x_m^{*2})^3}+\frac{yr^{*2}d'\left(r^*\right)L'(y^*)}{\left[L(y^*)\right]^2},$$

$$B_3=\frac{d'\left(r^*\right)L'(y^*)y^*r^{*2}}{[L(y^*)]^2}\left(2\beta x_m^*+\frac{2cx_m^*y^*}{(1+e_1x_m^{*2})^2}\right)+\left(2\beta x_m^*+\frac{2cx_m^*y^*}{(1+e_1x_m^{*2})^2}\right)\left(-d(r^*)+\frac{fx_m^{*2}}{1+e_1x_m^{*2}}\right)\left(1-\frac{2r^*}{L(y^*)}\right)$$
$$-\left(1-\frac{2r^*}{L(y^*)}\right)\frac{2fcx_m^{*3}y^*}{(1+e_1x_m^{*2})^3},$$

$$B_4=-\alpha e^{-\gamma\tau}=\left(-\beta x_m^*-\frac{cx_m^*y^*}{(1+e_1x_m^{*2})^2}\right),$$

$$B_5=\alpha e^{-\gamma\tau}\left(-d\left(r^*\right)+1-\frac{2r^*}{L(y^*)}+\frac{fx_m^{*2}}{1+e_1x_m^{*2}}\right)=\left(\beta x_m^*+\frac{cx_m^*y^*}{(1+e_1x_m^{*2})^2}\right)\left(-d\left(r^*\right)+1-\frac{2r^*}{L(y^*)}+\frac{fx_m^{*2}}{1+e_1x_m^{*2}}\right),$$

and

$$B_6 = -\alpha e^{-\gamma\tau} \frac{d'\left(r^*\right)L'(y^*)yr^2}{[L'(y^*)]^2} = \left(-\beta x_m^* - \frac{cx_m^* y^*}{(1+\mathrm{e}_1\, x_m^{*\,2})^2}\right)\frac{d'\left(r^*\right)L'(y^*)y^* r^{*2}}{[L'(y^*)]^2},$$

To show the positive equilibrium $E^*(\mathrm{x}_m^*, \mathrm{y}^*, \mathrm{r}^*)$ is locally asymptotical stable for all $\tau \geq 0$, we use the following theorem [16].

Theorem 4.1: A necessary and sufficient condition for $E^*(\mathrm{x}_m^*, \mathrm{y}^*, \mathrm{r}^*)$ is locally asymptotically stable for $\tau \geq 0$ is,

(I) The real parts of all roots of $\phi(\lambda, 0) = 0$ are negative.

(II) For all real b and $\tau \geq 0$, $\phi(ib, \tau) \neq 0$, where $i = \sqrt{-1}$.

Theorem 4.2: Then the positive equilibrium point $E^*(\mathrm{x}_m^*, \mathrm{y}^*, \mathrm{r}^*)$ for the system (2.4) is locally asymptotically stable providing following conditions,

(i) $e_1 d(\mathrm{r}) > \dfrac{f}{e_1 x_m + 2}$,

(ii) $e_1 d\left(r\right) > 1$.

Proof: Prove of this theorem is given in two steps as follow,

Step-I. Substituting $\tau = 0$ in equation (4.1), we obtain,

$\lambda^3 + (B_1 + B_4)\lambda^2 + (B_2 + B_5)\lambda + (B_3 + B_6) = 0 = \phi(\lambda, 0)$, (4.2) $(B_1 + B_4) > 0$, $(B_2 + B_5) > 0$ and $\{(B_1 B_2 + B_4 B_2 + B_1 B_5 + B_4 B_5) - (B_3 + B_6)\} > 0$.

Then by Routh-Hurwitz criterion (4.2) has all roots are negative then in the absent of delay, $E^*(\mathrm{x}_m^*, \mathrm{y}^*, \mathrm{r}^*)$ is asymptotically stable.

Step-II. In this step we show that $\phi(\lambda, \tau) \neq 0$ for $\lambda = ib$, for real b ,

Let $b = 0$ then

$\phi(0, \tau) = B_3 + B_6 \neq 0$, it is hold II condition.

If $b \neq 0$ then equation (4.1) can be written as:

$$-ib^3 - iB_1 b^2 + iB_2 b + B_3 + e^{-ib\tau}\left\{-B_4 b^2 + iB_5 b + B_6\right\} = \phi(ib, \tau),$$

Now equating real and imaginary parts of this equation we obtain the following equations,

$$-(B_4 b^2 - B_6)\sin b\tau + bB_5 \cos b\tau = b^3 - bB_2 \tag{4.3}$$

$$(B_4 b^2 - B_6)\cos b\tau + bB_5 \sin b\tau = b^2 B_1 - B_3 \tag{4.4}$$

After squaring and adding the above equations we get the equation,

$b^6 + b^4(-2B_2 + B_1^2 - B_4^2) + b^2(B_2^2 - 2B_1B_3 + 2B_4B_6 - B_5^2) + (B_3^2 - B_6^2) > 0$, as

$$B_1^2 - B_4^2 - 2B_2 = 3\left\{\beta x_m + \frac{cx_m y}{(1+e_1x_m^2)^2} + d(r) - \frac{fx_m}{(1+e_1x_m^2)}\right\}^2 + \left\{4\left(\frac{2r}{L(y)} - 1\right)\left[\frac{fx_m^2 y}{(1+e_1x_m^2)^3} + \beta x_m + cx_m y\right] - \frac{yr^2 d'(r) L'(r)}{[L(y)]^2}\right\} + \left\{1 + \frac{2r}{L(y)} + 2d(r) - \frac{2fx_m^2}{(1+e_1x_m^2)^2}\right\}\left\{\left(\frac{2r}{L(y)} - 1\right) + 2\left(-d(r) + \frac{fx_m^2}{1+e_1x_m^2}\right)\right\}$$

$B_1^2 - B_4^2 - 2B_2 > 0$, if $e_1 d(r) > \dfrac{f}{e_1 x_m + 2}$, condition (i)

$B_1 - B_5 > 0$, $B_4B_6 - B_1B_3 > 0$ then $B_2^2 - B_5^2 + 2(B_4B_6 - B_1B_3) > 0$ and

$$B_3^2 - B_6^2 = \left(1 - \frac{2r}{L(y)}\right)^2\left(-2\beta d(r)x_m - \frac{2cx_m yd(r)}{(1+e_1x_m^2)^2} + \frac{2\beta x_m^3}{1+e_1x_m^2}\right)^2 + \frac{8yr^2 d'(r) L'(r)}{[L(y)]^2}\left(\beta x_m + \frac{cx_m y}{(1+e_1x_m^2)^2}\right)\left(\frac{2r}{L(y)} - 1\right)\left(\frac{cx_m yd(r)}{(1+e_1x_m^2)^2} + \beta d(r)x_m - \frac{\beta x_m^3}{1+e_1x_m^2}\right)$$

then $B_3^2 - B_6^2 > 0$ if $\left(\dfrac{cx_m yd(r)}{(1+e_1x_m^2)^2} + \beta d(r)x_m - \dfrac{\beta x_m^3}{1+e_1x_m^2}\right) > 0$ after solving this equation we get $e_1 d(r) > 1$, condition (ii)

So $B_1^2 - B_4^2 - 2B_2 > 0$, $B_2^2 - 2B_1B_3 + 2B_4B_6 - B_5^2 > 0$ and $B_3^2 - B_6^2 > 0$.

Hence $\phi(ib,\tau) \neq 0$ for real b. Therefore the unique positive equilibrium $E^*(x_m^*, y^*, r^*)$ is locally asymptotically stable for all $\tau \geq 0$ and the delay is harmless in this case.

5. BIFURCATION ANALYSIS

The characteristics equation of the system (2.4) for the equilibrium point $E^*(x_m^*, y^*, r^*)$ is given as,

$$\varphi(\lambda,\tau) = \lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 + (B_4\lambda^2 + B_5\lambda + B_6)e^{-\lambda\tau} = 0.$$

Let the eigenvalue of equation is in the form of $\lambda = a(\tau) + ib(\tau)$ and function of τ . Differentiating (4.1) with respect to τ , we obtain the following equation

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = -\left(\frac{3\lambda^2 + 2\lambda B_1 + B_2}{\lambda(\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3)}\right) + \left(\frac{3\lambda B_4 + B_5}{\lambda(B_4\lambda^2 + B_5\lambda + B_6)}\right) - \frac{\tau}{\lambda}, \quad (5.1)$$

If λ has only purely imaginary part $a(\tau) = 0$ and $b(\tau) \neq 0$ then $\lambda = ib(\tau)$ equation (5.1) become,

$$\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1} = \left(\frac{B_2^2 - 4b^2B_2 + 3b^4 - 2B_1^2b^2 + 2B_1B_3}{(b^3 - bB_2)^2 - (b^2B_1 - B_3)^2}\right) - \left(\frac{B_5^2 b + 3B_6B_4 - 3B_4^2b^2}{B_5^2b^2 - (B_6 - B_4b^2)^2}\right)$$

At $\tau = \tau_c$ $\left[\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\tau=\tau_c} \neq 0$, it is verified by numerically then it is hold transversality conditions.

The value of delay $\tau = \tau_c$ is given as,

$\tau_c = \frac{1}{b}\arccos\left\{\frac{b^4(B_5 + B_1B_4) - b^2(B_1B_6 + B_4B_3 + B_2B_5) + B_6B_3}{B_5{}^2b^2 + (B_4b^2 + B_6)^2}\right\} + \frac{2k\pi}{b}$, $k = 0,1,2.....$ this value obtains by the equations (4.3) and (4.4).

6. PERSISTENCE OF THE SYSTEM

Biologically, persistence means the survival of all populations in future time. Mathematically, persistence of a system means that strictly positive solutions do not have omega limit points on the boundary of a non-negative cone. The persistence of the system (2.4) is give by following theorem,

Theorem 6.1: Assuming that $\alpha e^{-\nu\tau} > c$, the permanence of the solution of the system (2.4) is given by the following conditions (6.1), (6.2), (6.3).

Proof: From first equation of the (2.4),

$$\frac{dx_m}{dt} = \alpha e^{-\nu\tau}x_m - \beta x_m^2 - cx_m\left(\frac{x_m y}{1+e_1x_m^2}\right)$$

$$\frac{dx_m}{dt} \geq \alpha e^{-\nu\tau}x_m - \beta x_m^2 - cx_m$$

$$\frac{dx_m}{dt} \geq (\alpha e^{-\nu\tau} - \beta x_m - c)x_m ,$$

Then $\liminf_{t\to\infty} x_m \geq \frac{\alpha e^{-\nu\tau} - c}{\beta} > 0$ if $\alpha e^{-\nu\tau} > c$ (6.1)

Second equation of the system (2.4),

$$\frac{dy}{dt} = -d\left(r\right)y + \frac{fx_m^2 y}{1+e_1x_m^2}$$

Let $\limsup_{t\to\infty} \mathrm{y} \leq \eta_0$ and $\liminf_{t\to\infty} \mathrm{x}_m \geq \frac{\alpha e^{-\upsilon\tau} - c}{\beta} = \eta_1$

Then above equation can be written as,

$\frac{dy}{dt} \geq \frac{f\eta_0\eta_1}{1+e_1\eta_1^2} - d_0 y$, then we get

$$\liminf_{t\to\infty} \mathrm{y} \geq \frac{f\eta_0\eta_1}{(1+e_1\eta_1^2)d_0} = \frac{f\eta_0(\alpha\,\mathrm{e}^{-\upsilon\tau} - \mathrm{c})\beta}{\left[\beta^2 + e_1(\alpha e^{-\upsilon\tau} - c)^2\right]d_0} \quad (6.2)$$

Now third equation of system (2.4)

$$\frac{dr}{dt} = r\left(1 - \frac{r}{L(y)}\right)$$

$$\frac{dr}{dt} \geq r\left(1 - \frac{r}{L_0 - L_1}\right) \text{,after solving, we obtain } \liminf_{t\to\infty} r \geq L_0 - L_1 \qquad (6.3)$$

7. NUMERICAL SIMULATION AND DISCUSSION

Analytical studies always remain incomplete without numerical verification of the results. To facilitate the interpretation of our mathematical findings by numerical simulations, we assume

$d(r) = d_0 - d_1 + d_1 e^{-r}$, $d_1 \in (0, d_0)$, And consider the set of parameter values as

$\alpha = 0.8, \beta = 0.3, \gamma = 0.1, \tau = 10, c = 4.5, f = 10, d_0 = 0.4, d_1 = 0.3, L_0 = 10, L_1 = 2, e_1 = 0.5$. For the above set of parameter values, the equilibrium $E^*(0.1003, 0.5297, 9.1788)$ is obtained. By using these parameters the following figure is given as,

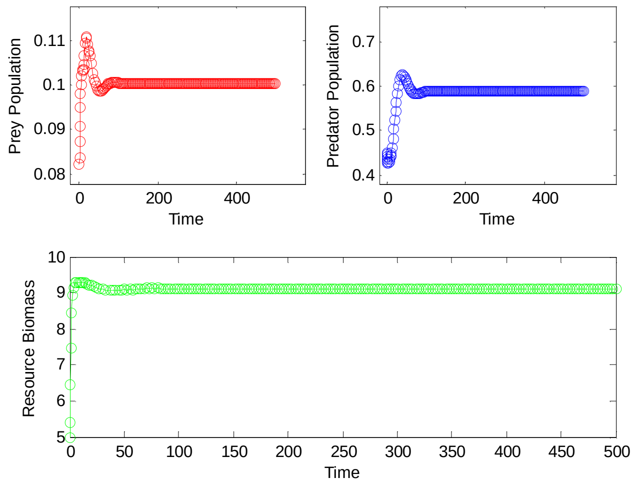

Figure 1: Variation of Prey Population, Predator Population and Resource Biomass with Respect to Time

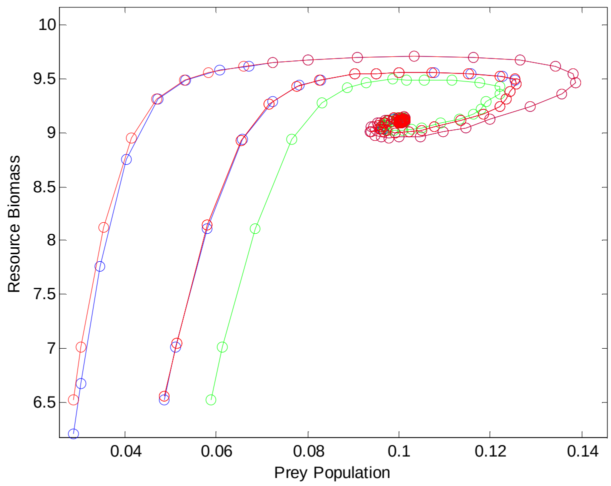

Figure 2: Variation of Prey Population With Respect to Resource Biomass at Different Initials Points

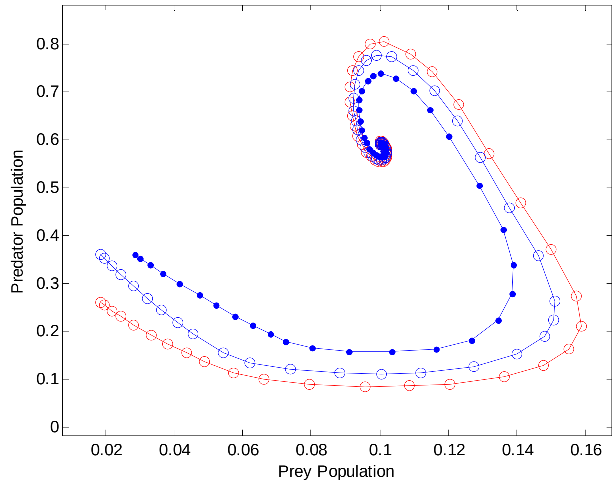

Figure 3: Variation of Prey Population with Respect to Predator Population at Different Initials Points

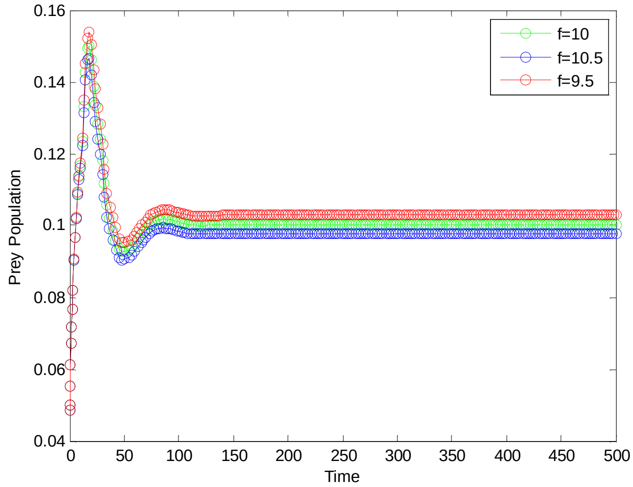

Figure 4: Variation of Prey Population with Respect to Time at Different Values of Conversion Rate

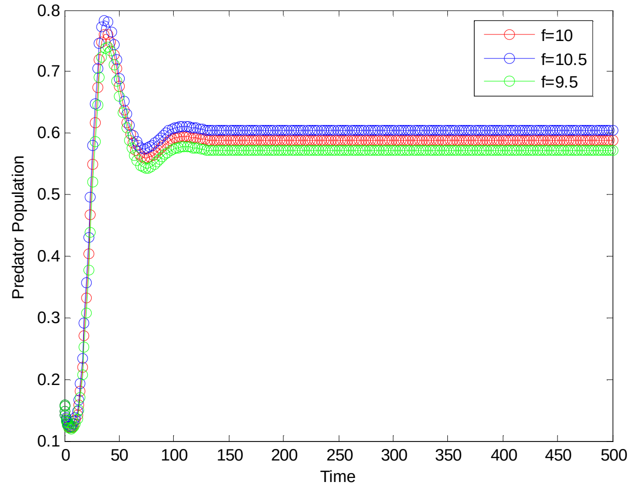

Figure 5: Variation of Predator Population with Respect to Time at Different Values of Conversion Rate

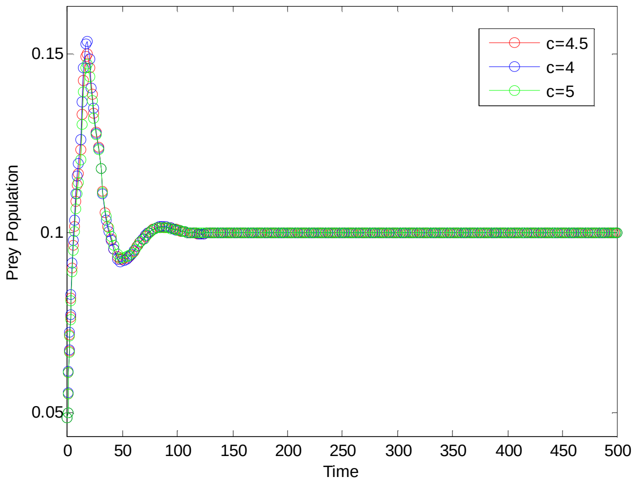

Figure 6: Variation of Prey Population with Respect to Time at Different Values of Capturing Rate of Predator

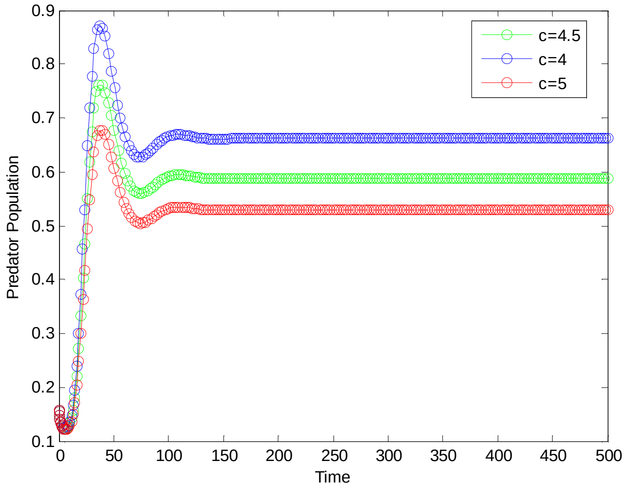

Figure 7: Variation of Predator Population with Respect to Time at Different Values of Capturing Rate of Predator

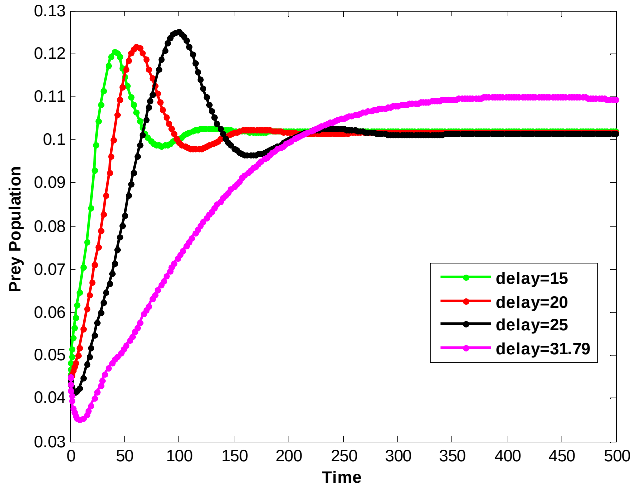

Figure 8: Variation of Prey Population with Different Values of Delay

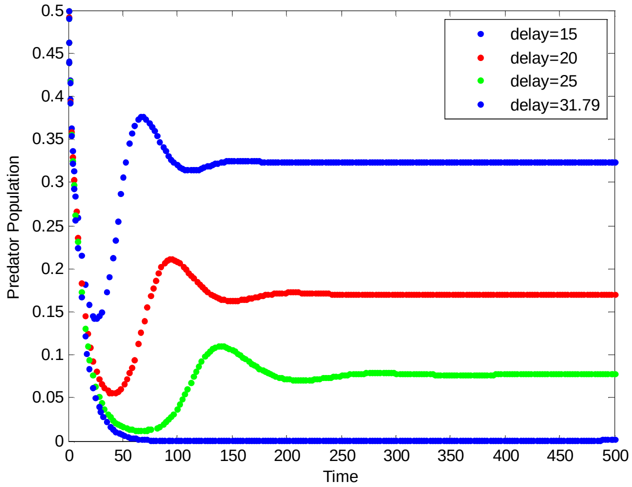

Figure 9: Variation of Predator Population with Different Values of Delay

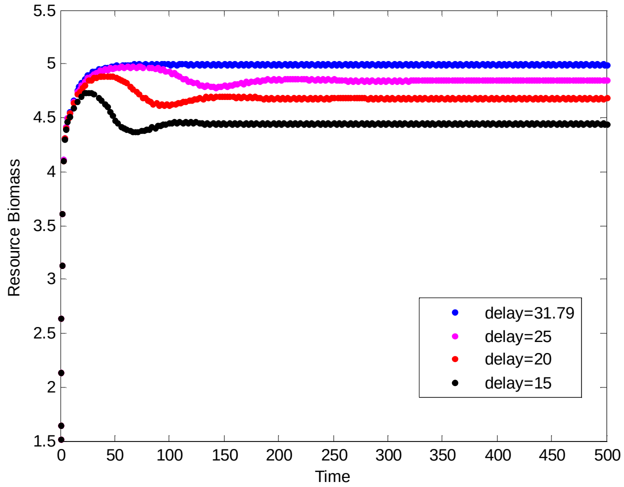

Figure: 10. Variation of Resource Biomass with Different Values of Delay

Figure 1 shows the behavior of the prey, predator population and resource biomass with respect to time, we obtain that after some time pre, predator and resource biomass being constant that means they got their equilibrium points. In Figure 2 discuss the behavior of prey and resource biomass with different initial points of them, here we see that it converge at their equilibriums point .In same manner in Figure 3 we see the variation between the prey and predator population at different initial points, we get that they converge toward their equilibrium point.

Figure 4 and 5 describe the effect on the prey and predator population with different values of conversion rate of prey. As conversion rate increase the prey population decrease and predator population increase.

In Figure 6 and 7 we see the variation of prey and predator population with change of capturing rate of predator, there is no change of equilibrium point of prey population but at initially it increase as capturing rate decrease, on the other hand predator population decrease as the capturing rate increase .

Figure 8 described the behavior of the prey population with different value of delays, as value of delay increased the population of prey does not change at initially it decreased but in Figure 9 predator population decrease as value of delay increases but in figure 10 the value of resource biomass does not change with change of delays value.

In the absent of resource biomass behavior of the system can be seen as follow by figure and table.

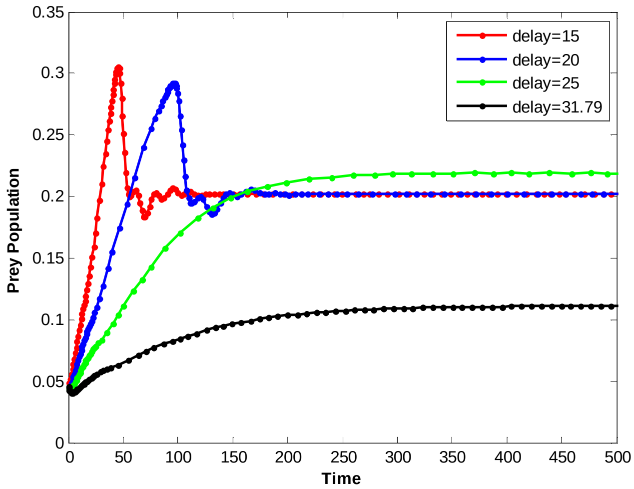

Figure 11: Variation of Prey Population with Different Values of Delay in Absence of Resource Biomass

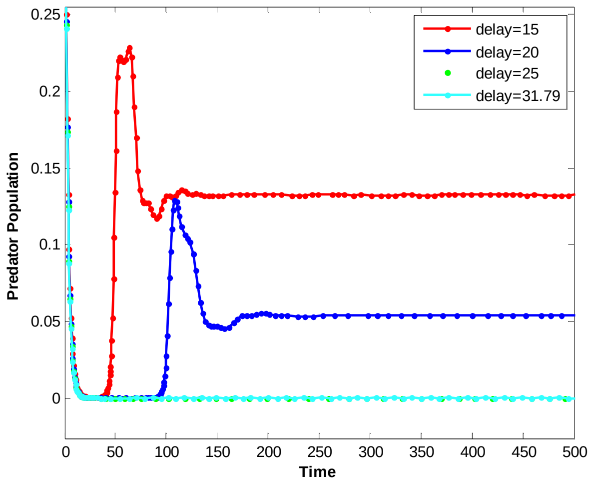

Figure 12: Variation of Prey Population with Different Values of Delay in Absence of Resource Biomass

Table 1: Absence of Resource Biomass

τ	x_m	y
10	0.2022	0.263
15	0.2022	0.1325
20	0.2022	0.05354
25	0.2022	0.005567
31.79	0.111	0.0000

Now from Figure 11-12 we obtained that as the value of delay increases value of prey population at initially does not change but as $\tau > 25$ value of prey slightly decreases, but in predator population decreases as delay value increases.

CONCLUSIONS

In this paper we studied a prey-predator-resource biomass mathematical model where prey species are delayed. Delay of prey is taken as the time of maturation period of prey, predator does not interact with immature prey. After numerical simulation we obtain following results,

In this paper by numerical simulation for parameters value system is stable in certain conditions, here we see that as conversion rate of predator is increases prey population decreases but predator population increases in the same manner see the effect of capturing rate of Predator on prey and predator population.

Here we see the behavior of the system at values of delay, as in paper of Agarwal and Devi [2], they describe that as value of delay changes effect of delay on resource biomass is negligible but in this paper's model we use the holling type III functional response as the present of this term we see as the value of delay change value of resource biomass also change.

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